Critique: Does the small scale inhomogeneity affect the large scale cosmology?

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Abstract

A critique is given here based on two papers debating the backreaction effect in large scale cosmology. One paper hopes to prove the backreaction is irrelevant based on analogy analysis and theoretical framework. It is criticised by the other paper which points out the flaws in its approach. We show that the latter paper is more rigorous and convincing.

1 Introduction

It is widely accepted that the FLRW(Friedmann-Lemaitre-Robertson-Walker) model well describes our universe on large scale. Combining with the resulting ΛCDM model, It explains many phenomena successfully in observation, like CMB and galactic dynamics. The FLRW model, therefore, is regarded as one of the foundation of cosmology.

However, questions could be posed on its validity, especially on the assumption it makes. To be able to apply the FLRW model to a real, inhomogeneous universe, such as our own, one assumes that: (a) the deviation of the real metric from a FLRW one is quite small in all scales, except in region close to the compact object like black holes. (b) the nonlinear terms arising from the differences between the real and FLRW metric are negligible compared to the linear terms in Einstein equation. Then the deviation of the real cosmological dynamics from the FLRW cosmology could be described by the standard perturbation theory. While (a) is supported by some estimates (like in Holz and Wald, 1998), (b) is not secured, because those non-linear terms are directly linked to the differences between the real density distribution and the background constant density in FLRW model, and these differences are considerably large especially on small scales. In this case, these terms from inhomogeneities can not be ignored in the dynamics equations, affecting the cosmology evolution. Their effect is the so-called "backreaction".

The question above could be expressed in another way, as it is given in Clarkson (2011): The standard way to deal with the small scale inhomogeneity in cosmology is working with the perturbation on a homogeneous and isotropic "background", and the evolution of the "background" is derived directly from the field equation with this homogeneous and isotropic metric. However, given the field equation is describing the spacetime itself, it is more reasonable to start with an inhomogeneous metric considering the exact mass distribution and then smooth it to large scale. The question is, will this process give the same background dynamics as the "standard" treatment? The nonlinearity of the field equation indicates that it will not.

However, the more relevant question is whether it is important for cosmology, and opinions of cosmologists for this vary. While some papers (e.g. Hirata and Seljak, 2005) claim that the backreaction is entirely negligible and supported by the success of the FLRW (and perturbation) models, some other physicists believe the backreaction is strong enough and could be a possible explanation for "dark energy" effect, i.e. the accelerating expansion of the universe, without introducing a cosmological constant, new matters or modified gravity(Buchert and Carfora, 2003; Coley, Pelavas, and Zalaletdinov, 2005; Kolb, Marra, and Matarrese, 2008).

The backreaction serves as the "dark energy" is a quite attractive proposal, as it solves new problems in the old theoretical frame. However, it is far from being accepted by the field. Many papers investigating backreaction effect in cosmology have been published through the years, and the debate is still going on nowadays (Clifton and Sussman, 2019).

In this report, we give a critique on the papers of Green and Wald (2014) *"how well is our Universe described by an FLRW model?"* and Buchert et al. (2015) *"Is there proof that backreaction of inhomogeneities is irrelevant in cosmology?"* in the centre of this debate. We are going to refer authors of these two papers by abbreviation GW and Buchert respectively in the rest of this report.

This critique is organised as below: In section 2, we present the main methodologies and conclusions of the two papers. In section 3, we provide an evaluation and discussion, and Our conclusion is given in section 4. Our notion for this report follows GW's paper with $c = G = 1$, and the usual Einstein summation convention is used. The Roman letters (a,b,c,...) from the early part of alphabet denote spacetime indices from 0 to 3 (or 0 to 2 in a two-dimensional universe).

2 Summary

2.1 Green and Wald, 2014

As we mentioned in the introduction part, there are two main problems that FLRW model supporters need to solve: (i) Is it reasonable to assume a homogeneous and isotropic background directly, without any mathematical treatment, to represent the large scale evolution of our universe? (ii) Is the nonlinear contribution arising from the small scale inhomogeneity really negligible? If not, what could the influence of this backreaction be?

In GW's paper, these two problems are solved separately. For the first problem, A two-dimensional analogy is proposed, as shown in figure 1. The authors consider a convex polyhedron with many faces (e.g. 30,000 faces), and discuss what the cosmologists among the creatures living on the surface will do to find a background metric for their two-dimensional "universe". As the authors claim, the creatures could do various "astronomical observations" and find that, while the geometry and curvature vary enormously on small scales, the measurement in large scale is agreed satisfactorily with a spherical metric. They may also find that there are some large scale observations are against the spherical metric, such as it is possible to have two different "geodesics" between two same points even they are not far enough to be seen as antipodal on a sphere. As a result, the two-dimensional cosmologists could conclude that the sphere metric provides a good approximation as a homogeneous and isotropic background, though it may have "some significant deviations" in large scale behaviour and provides an inferior description on small scales.

The point, the authors claims, is that what we human did to find the FLRW metric is in a precisely similar manner to the two-dimensional example. With this analogy, their paper says the first problem is "trivially solved": Not only that we conclude the FLRW metric based on similar varied observations, but the concluded metrics also have similar properties: in both cases, the exact metric could be written as $g_{ab} = g_{ab}^{(0)} + \gamma_{ab}$, where the $g_{ab}^{(0)}$ is the background and γ_{ab} is much smaller than it. However, the derivative of the γ_{ab} is not necessarily small,

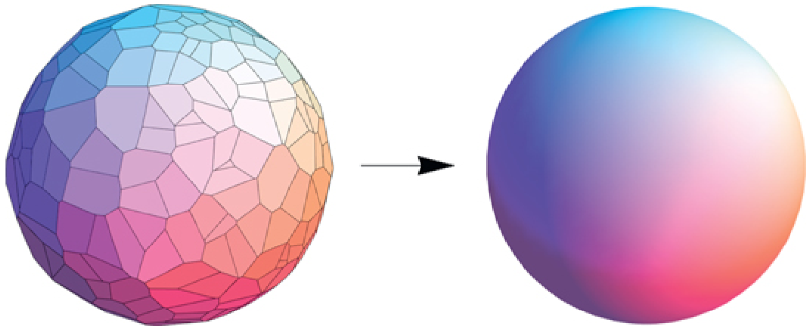

Figure 1: The analogy provided in Green and Wald (2014). The authors suggest that what we did to conclude a FLRW model for our universe background is just like what the two-dimensional creature on a polyhedron did to conclude a spherical geometry for their "universe" background.

and the second derivative of γ_{ab} could even be substantial. These settings correspond to the strong matter inhomogeneities in the real universe, the source of the backreaction.

To solve the second problem, the authors investigate the "exact" Einstein equation based on the assumptions above. They show that even the non-linear terms from the large derivatives are included, they would only behave as some kind of radiation. Mathematically speaking, they prove a theorem that the backreaction term $t^{(0)}_{ab}$ in the equation

$$G_{ab}\left(g^{(0)}\right) + \Lambda g^{(0)}_{ab} = 8\pi\left(T^{(0)}_{ab} + t^{(0)}_{ab}\right) \tag{1}$$

satisfies two conditions: (i) traceless, ${t^{(0)}}^{a}{}_{a} = 0$. (ii) weak energy condition, i.e., $t^{(0)}_{ab}t^a t^b \geq 0$ with any timelike vector t^a with respect to $g^{(0)}_{ab}$. Combined with the FLRW symmetry, These two conditions result in a $P = \frac{1}{3}\rho$ fluid of normal radiation. So it is impossible for matter inhomogeneities to contribute the backreaction, nor achieve the dark energy effect. Therefore, the second problem is solved as well.

To prove the theorem of the two conditions (i) (ii) they give, the authors propose a new framework similar to the perturbation using an "infinitesimal" γ_{ab}. They construct a one-parameter metric family $g_{ab}(\lambda)$, such that $\left[g_{ab}(\lambda) - g^{(0)}_{ab}\right] \to 0$ when $\lambda \to 0$. $g^{(0)}_{ab} \equiv g_{ab}(0)$ is the background. Then they translate their assumptions of the metric into the following four points:

1. for all $\lambda > 0$, the "exact" Einstein equation holds, i.e.,

$$G_{ab}(g(\lambda)) + \Lambda g_{ab}(\lambda) = 8\pi T_{ab}(\lambda) \tag{2}$$

   and $T_{ab}(\lambda)$ satisfies weak energy condition (normal matter).

2. There exists a smooth positive function $C_1(x)$ in spacetime such that

$$|\gamma_{ab}(\lambda, x)| \leq \lambda C_1(x) \tag{3}$$

   Here the authors define $\gamma_{ab}(\lambda, x) \equiv g_{ab}(\lambda, x) - g_{ab}(0, x)$.

3. There exists a smooth positive function $C_2(x)$ in spacetime such that

$$|\nabla_c \gamma_{ab}(\lambda, x)| \leq C_2(x) \tag{4}$$

4. There exists a smooth tensor field μ_{abcdef} in spacetime such that

$$\mathrm{w}-\lim_{\lambda\to 0}[\nabla_a\gamma_{cd}(\lambda)\nabla_b\gamma_{ef}(\lambda)] = \mu_{abcdef}. \tag{5}$$

Here the authors introduce the concept of weak limit (notated as $\mathrm{w}-\lim$) as appearing in assumption 4. The weak limit of a tensor $A_{a_1\ldots a_n}(\lambda)$ is defined as the $A^{(0)}_{a_1\ldots a_n}$ which satisfies the following equation given an arbitrary smooth tensor field $f^{a_1\cdots a_n}$.

$$\mathrm{w}-\lim_{\lambda\to 0}[A_{a_1\ldots a_n}(\lambda)] = A^{(0)}_{a_1\ldots a_n} \;\Leftrightarrow\; \lim_{\lambda\to 0}\int f^{a_1\cdots a_n}A_{a_1\ldots a_n}(\lambda) = \int f^{a_1\cdots a_n}A^{(0)}_{a_1\ldots a_n} \tag{6}$$

They claim that it corresponds roughly to take the local spacetime average and then take the limit of $\lambda \to 0$. With these four assumptions, the authors manage to prove the two conditions above. The detailed derivation of this is provided in another paper of the authors (Green and Wald, 2011).

After answering the two questions, the authors also criticise some other approaches leading to the conclusion of a strong backreaction from matter inhomogeneities. They discuss the drawbacks of these approaches and point out that they provide an inadequate description of the real universe. In the final part of the paper, the authors claim that with a good "dictionary" mapping the quantities between linearised Newtonian gravity and linearised general relativity, the leading (linear) term of the inhomogeneity backreaction could be approximated by Newtonian gravity "excellently". Since these two parts are mainly the secondary arguments, we do not present them in detail here.

To sum up, Green and Wald demonstrate in their paper that it is reasonable for us to conclude the FLRW model directly, and it describes our universe "extremely well". The linear terms of backreaction could be represented via standard Newtonian perturbation, and the nonlinear terms behave like radiation. Therefore, a strong backreaction is unlikely to be from the matter inhomogeneities, or replace the dark energy as some authors suggest.

2.2 Buchert et al., 2015

One year later, Buchert and his collaborators published the paper against the proof of GW about the negligible backreation effects in cosmology. They organise the paper nearly in line with the presentation of the GW's paper questioning the validity of their framework, and claims that GW have not solved the two problems convincingly.

The paper starts with a full analysis of the two-dimensional analogy given by GW. Buchert suggests that GW's answer stating that the fitting problem is trivially solved is "dictated by common sense" thus unreliable.

Firstly, GW construct a convexity polyhedron for an "apparently" similarity to the sphere geometry. However, The convexity is a characteristic that is only available when embedded into a higher dimensional space, i.e. is linked to the extrinsic curvature, rather than the intrinsic curvature we need to deal with in real cosmological problems. So the intuitive similarity in three-dimensional view is irrelevant to the real problem. On the other hand, while the "astronomical observation" could measure the intrinsic curvature and its derivatives, one can not specify the metric uniquely based on this information. Buchert says that looking for a best-fitting metric from the curvature information is still a problem under investigation.

Buchert then gives a further explanation on the difficulty to find a proper metric for the cosmologists on the two-dimensional surface. The polyhedron only has nonzero curvatures on its vertices, and it is "almost-everywhere flat". By a proper construction (using a dual metrical triangulation), it is easy to find a polyhedron has thousands of flat vertices and only a few others that contain curvatures. In this case, even based on the large scale observation, the two-dimensional creature may finally conclude that the universe they live is flat, even though the actual background geometry is spherical. They may consider the small defect in curvature in observation could be ignored or averaged to zero on large scales, leading them to an incorrect background metric. Nevertheless, If the creatures could measure over the whole polyhedron surface, they would find the curvature defects they come across can not be averaged out to zero on a flat background metric, which is as expected because the actual geometry of their universe is spherical. Actually, it is also the only way for the creatures to find the correct geometry for their universe, no matter whether the coordinate components of their metric is close to a spherical/flat metric or not.

GW believe the first problem is trivially solved by this analogy, as the two-dimensional creature would conclude a spherical background using ”cosmological principle” assuming a homogeneous and isotropic metric. However, based on the argument we summarised above, Buchert shows the creatures would conclude an incorrect flat background if they really rely on the ”cosmological principle”. This is precisely because they ignore the small scale inhomogeneities, i.e. the curvature defect on vertices. The analogy does not solve the first problem as GW expected; on the contrary, it highlights the problem faced by the FLRW model.

After remarking several papers discussing mathematically rigorous mapping between polyhedron and sphere metric, Buchert moves to discuss the GW's solution to the second problem, i.e. their approach of analysing backreaction. As we present briefly in section 2.1, GW develop their framework on the four assumptions and weak limit scheme. They suggest that the latter is roughly a local space average. However, Buchert claims that both their assumptions and weak limit scheme are problematic and irrelevant to real physical backreaction.

GW construct a one-parameter family of metric $g_{ab}(\lambda)$ where a decreasing λ represents the smoothing process from a ”very” inhomogeneous metric $g_{ab}(1)$ which describes the real universe, to the homogeneous background $g_{ab}(0)$. In the assumption 1 they ask the exact Einstein equation holds for all non-zero λ:

$$G_{ab}(g(\lambda)) + \Lambda g_{ab}(\lambda) = 8\pi T_{ab}(\lambda) \quad ; \quad \lambda \in (0, 1] \tag{7}$$

and in the limit $\lambda \to 0$ the backreaction appears.

$$G_{ab}(g(0)) + \Lambda g_{ab}(0) = 8\pi \left(T_{ab}(0) + t_{ab}(0) \right) \tag{8}$$

However, as pointed out by Buchert, the smoothing and non-linear operation such as taking derivative do not commute. Therefore, the Einstein equation should not hold for any $\lambda \neq 1$. The backreaction term $t^{(0)}_{ab}$ in GW framework jump out from nothing when the $\lambda \to 0$, and it could only represent a part of the real backreaction term. The correct form of the Einsteins equation should be

$$G_{ab}(g(\lambda)) + \Lambda g_{ab}(\lambda) = 8\pi \left(T_{ab}(\lambda) + \tau_{ab}(\lambda) \right) \quad ; \quad \lambda \in (0, 1) \tag{9}$$

and the limit should give

$$G_{ab}(g(0)) + \Lambda g(0) = 8\pi \left(T_{ab}(0) + t_{ab}(0) + w - \lim_{\lambda \to 0} \left[\tau_{ab}(\lambda) \right] \right) \tag{10}$$

The extra λ-dependent term is not traceless in general. Moreover, Buchert claims that the assumption 4 of GW is too restrictive and requires unnecessary good behaviour of the product $\nabla\gamma(\lambda)\nabla\gamma(\lambda)$ as $\lambda \to 0$.

Buchert then criticises the weak limit scheme of GW for the averaging. Besides such kind of average is not in the usual average sense we use in discussing backreaction, He also highlights that such limitation process gives not functions but distributions, whose geometrical and physical meanings are more subtle and harder to interpret. Furthermore, overlooking the distribution nature of the curvature could lead us to a wrong fitting, just like overlooking the curvature defect on a polyhedron leads to an incorrect flat background. Therefore, GW's proof for a traceless, normal radiation-like effective tensor is unreliable.

Several examples demonstrate the problems in GW's assumptions and scheme are given in the following section of Buchert's paper. Then he replies to the objection raised in GW's paper on strong-backreaction approach and point out that it is impossible to mimic certain GR phenomena using Newtonian approximation. As done for the last section, we are not going to present these supporting arguments in detail here.

In the final part of the paper, Buchert emphasise the importance of reconsidering our averaged background model in the precision cosmology era. Though the FLRW metric (maybe with backreaction correction) could still describe our universe in large scale satisfactorily, it should be backed by a concrete mathematical averaging process, rather than being treated as *a pirori* assumption.

3 Evaluation and Discussion

GW start their paper with a quite general introduction in backreaction and refer review articles of others to introduce the fitting problem in cosmology. Based on this, the authors summarise the two problems that they are going to solve in the paper, which provides a good map through the whole article.

However, as being pointed out by Buchert's paper, GW solve the first problem rather poorly. They hope an intuitive analogy could show the validity of assuming a homogeneous and isotropic metric from observation, on the contrary, this analogy shows such assumption is unreliable because it will lead to a wrong conclusion in the choice of background. What is more, even we accept it as a reasonable analogy to illustrate the similar situation in our universe, GW do not provide a convincing explanation why the two-dimensional creatures would conclude the specific properties (see the third paragraph, section 2.1) of their metric, which becomes their fundamental assumption in proof of the second problem.

Similar flaws also appear in GW's approach dealing with the backreaction. While the assumptions they would like to conclude from the analogy is somehow plausible, the weak-limit scheme they propose to represent spatial averaged properties is not straightforward as GW claims. A detailed examination of why it "roughly corresponds to a local space average" is needed but absent in GW's paper. Another problem is that the limit of the one-parameter family of the metric is heavily coordinate-dependent, as shown in examples of section 4 in Buchert's paper. GW do not discuss this issue in their paper while giving a robust coordinate-independent theorem. The backreaction also only appears when taking the limit, which is against the realistic smoothing process.

In the following section, GW provide a simple analysis of other approaches to strong backreaction and successfully demonstrate the hypersurface-dependent issues the spatial average scheme is facing via good examples. Then the authors proposed the methods to map GR backreaction to Newtonian approximation.

GW finally reach their conclusion backing the current FLRW model: The matter inhomogeneity does not affect the large scale cosmology. This argument is not so reliable because of the ill-defined and unphysical assumptions and scheme we mentioned above.

As a critic paper, Buchert only invokes a small introduction of backreaction in the first section. Then he summarises GW's main idea briefly, and shows that he is going to reply to GW's paper nearly point to point: the analogy for the first problem, the framework for the second problem, and other supporting arguments. The general outline of Buchert's paper is thus pretty clear.

Despite the clear big picture of the whole paper, Buchert presents his argument in a confusing way when it comes to particular points. The lack of illustrating figures for text also makes the paper harder to follow.

The second section of the paper is the criticism of the polyhedron analogy, and Buchert proposes several reasons supporting his argument. These reasons are not organised in good logical order, mixed with author's quoting on GW's original statement. The conclusion of this section also seems to be a new argument, rather than a summary of the body part he writes. It thus becomes challenging for readers to get information from this section. On the other hand, The author does provide enough explanations on the flaws of the polyhedron analogy, as we present in section 2.2, as long as readers are not confused by his presentation.

The following section against GW's mathematical treatment is somehow better organised. However, the author writes a quite long section for this. Buchert starts with the discussion on the trace of effective tensor in Newtonian cosmology, which we suppose should be an additional subsection rather than a leading one. Then he briefly explains the idea GW used and points out the assumption 4 is too strong. After that, Buchert moves to criticise that GW's one-parameter family of metric approach results in distributions not functions and even we generalise GW's scheme to consider distributional curvature, the assumption 1 of GW still constructs a backreaction that jumps from nothing only when taking the limit. Based on the logical flow above, the author provides a quite rigorous arguing here, though he has to extend his writing to include all points above, sacrificing the readability.

After going through the supporting arguments (section 2.2), Buchert concludes that the Green and Wald have not convincingly proved that backreaction from the matter inhomogeneity is negligible in cosmology. And it is strongly supported by its arguments given in the content of his paper.

4 Conclusion

In this report, we provide a critique on the paper of Green and Wald (2014) and Buchert et al. (2015). The former one (GW's) claims that the backreaction is negligible based on its theoretical framework. However, as we have shown in the content above, this idea is not rigorous and convincing because of the problematic analogy argument and ill-concluded assumptions and approaches.

On the other hand, the critic paper (Buchert et al., 2015) accomplishes its goal against GW's paper, roughly speaking. While the arguments presentations are expected to be more transparent and neater, Buchert manages to show readers that the proof of GW is unreliable based on reasonable explanations. Therefore, we would prefer to say this paper is more worthy of reference.

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